### § 8. *Summary.*

An integral equation is obtained for the density of radiant energy in radiative equilibrium and solved by successive approximation. The various approximations involve an arbitrary constant, which can be chosen in various ways so as to improve the accuracy of the solution. It is shown that the density-gradient taken with respect to the optical thickness $(dB(\tau)/d\tau)$ has a logarithmic infinity at the boundary, and approximations are obtained for the boundary temperature $T_0$ in terms of the effective temperature $T_1$. The temperature distribution thus found is applied to deduce the law of darkening of a stellar disc towards the limb; it is shown that the value $\frac{3}{5}$ for the coefficient of darkening is a better approximation than the Schwarzschild-Jeans value $\frac{2}{3}$; the distribution of intensity over the disc corresponds to a *mean* coefficient of about 0·61 or 0·62, but there is a more sudden drop in intensity very close to the limb.

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# *Radiative Equilibrium and Spectral Distribution.*

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(*Communicated by the Director, Solar Physics Observatory, Cambridge.*)

## § 1. *Introductory.*

In the investigation of radiative equilibrium it is necessary to exercise care in assuming that the radiation will possess certain of the properties of black-body radiation. For example, the density of radiant energy for isotropic black-body radiation is $aT^4$, where $a$ is Stefan's constant and T is the temperature; but it is not necessarily true that the energy-density in radiative equilibrium is equal to that of black-body radiation corresponding to the temperature of the matter at the point considered.

This point is of importance if one attempts to investigate the spectral distribution of the radiation.* The radiation differs from isotropic black-body radiation in that both the intensity and the spectral distribution vary from direction to direction, and the spectrum in any particular direction (for a given total intensity) will not in general be that of the corresponding black-body radia-

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* When writing this paper the author was unaware that the question of the spectral distribution of the light in radiative equilibrium had been previously discussed, but since the paper was finished his attention has been drawn to a paper by Lindblad, "The Distribution of Intensity in the Continuous Spectra of the Sun and Fixed Stars," *Uppsala Universitets Årsskrift*, 1, 1920, which discusses very fully the theoretical spectrum for radiative equilibrium and the variation of colour over the disc, with detailed comparison with the observations of Abbot, Fowle, and Aldrich. However, it seems worth while making the present communication, as the method of calculation and the method of expressing the results are somewhat different.

1921MNRAS..81..375M

tion. Further, the net flux, even if constant in amount, will in general vary in spectral distribution from place to place. Consider, for example, the radiation crossing a plane perpendicular to the direction of net flow, at a point where the temperature is T: that crossing from the cooler side to the hotter will be characteristic, on the whole, of some temperature lower than T, say $T_1$; that from the hotter side to the cooler of some temperature greater than T, say $T_2$. If $B_\lambda(T)d\lambda$ denotes the intensity of black-body radiation for temperature T between wave-lengths $\lambda$ and $\lambda + d\lambda$, the spectral distribution of the net flow of radiation is approximately

$$\{B_\lambda(T_2) - B_\lambda(T_1)\}d\lambda,$$

or, say,

$$(T_2 - T_1)\frac{dB_\lambda(T)}{dT}d\lambda,$$

which is a function of T and varies from point to point. The net flux is continually being transformed from shorter to longer wave-lengths as it advances in the direction of T decreasing. At the boundary of the material considered the net flux is radiated into space, and it is clear that its spectral distribution will not necessarily be that of black-body radiation of the same total intensity. The structure of the radiation as regards both its distribution in direction and its spectral distribution must be ascertained from detailed consideration of the laws of emission and absorption as expressed in the equations of transfer, with the appropriate boundary conditions.

It seems worth while working out this method in detail for the case of the outer layers of a star. We may suppose that almost the whole of the energy radiated by a star is derived from physical processes occurring not in the thin stratum that we see into, but in the star's interior; at any moment the state may be regarded as a steady one in which the amount of energy radiated externally is equal to that generated in the interior, so that the outer layers must take up such a state (as regards temperature-gradient, etc.) that they are capable of allowing the given amount of energy to be radiated through them. Given the total amount radiated per unit area, the spectral distribution of this radiation and the variation of this distribution with angle of emergence (*i.e.* variation of spectrum over the disc) should be determinate.

We will first adopt the simplified treatment in which the flow of radiation is "linear" or "tubular," the radiation being supposed confined to two streams, an inward and an outward. We will then modify the results by taking into account the spherical spread of the radiation. The first paragraph gives a few remarks on the general case when the absorption may depend on wave-length, but we shall mostly be concerned with the theoretical spectrum for the case when the coefficient of absorption is the same for all wave-lengths.

### § 2. *Selective Absorption and Radiative Equilibrium.*

We consider a medium stratified in planes perpendicular to the axis of $x$, extending indefinitely on the positive side of its boundary $x = 0$. The radiation is supposed to consist of two streams parallel to the axis of $x$; let $I_\lambda d\lambda$ be the intensity of the outward stream between wave-lengths $\lambda$ and $\lambda + d\lambda$ at the point $x$, where the temperature is T, $I_\lambda' d\lambda$ the inward stream. Let $B_\lambda d\lambda$ be the corresponding intensity of black-body radiation at the temperature T. As in the foregoing paper (§ 3), the intensities satisfy the equations

$$\frac{1}{2}\frac{dI_\lambda}{dx} = \rho k_\lambda (I_\lambda - B_\lambda), \qquad \frac{1}{2}\frac{dI_\lambda'}{dx} = \rho k_\lambda (B_\lambda - I_\lambda') \quad . \quad (1)$$

whilst the equality of the energy absorbed and emitted gives

$$2\int_0^\infty k_\lambda B_\lambda d\lambda = \int_0^\infty k_\lambda (I_\lambda + I_\lambda') d\lambda \quad . \quad . \quad . \quad (2)$$

Here $k_\lambda$ is the coefficient of absorption in the neighbourhood of $\lambda$. The outward net flux of energy is $\pi F_\lambda d\lambda$, where

$$F_\lambda = I_\lambda - I_\lambda' \quad . \quad . \quad . \quad . \quad (3)$$

We shall write

$$\int_0^\infty I_\lambda d\lambda = I, \quad \int_0^\infty I_\lambda' d\lambda = I', \quad \int_0^\infty B_\lambda d\lambda = B, \quad \int_0^\infty F_\lambda d\lambda = F.$$

From equations (1),

$$\frac{1}{2}\frac{dF_\lambda}{dx} = \frac{1}{2}\frac{d(I_\lambda - I_\lambda')}{dx} = \rho k_\lambda (I_\lambda + I_\lambda' - 2B_\lambda),$$

whence, integrating with respect to $\lambda$ and using (2),

$$\frac{dF}{dx} = 0, \qquad F = \text{constant},$$

as it should be. If we put

$$\Delta_\lambda = I_\lambda + I_\lambda' - 2B_\lambda,$$

then $2\pi\Delta_\lambda / c$, where $c$ is the velocity of light, is the excess of radiant energy per unit volume of wave-length $\lambda$ over that of black-body radiation for the temperature at the point $x$. We can now write equations (1) and (2) in the form

$$\left.\begin{aligned} \frac{1}{2}\frac{dF_\lambda}{dx} = \rho k_\lambda \Delta_\lambda, \qquad \frac{1}{2}\frac{d\Delta_\lambda}{dx} = \rho k_\lambda F_\lambda - \frac{dB_\lambda}{dx} \\ \int_0^\infty k_\lambda \Delta_\lambda d\lambda = 0. \end{aligned}\right\} \quad . \quad . \quad (4)$$

It is now clear that it is impossible for $\Delta_\lambda$ to be always zero; for this would imply, first, that $F_\lambda = \text{constant}$ (from place to place), secondly, that $k_\lambda F_\lambda \propto dB_\lambda / dT$, as far as variation with $\lambda$ is concerned; but since $dB_\lambda / dT$ as a function of $\lambda$ alters with T and

1921MNRAS..81..375M

therefore from place to place, these two results are contradictory. Further, unless $k_\lambda$ is independent of $\lambda$ the *total* excess of energy $\Delta$ (*i.e.* $\int \Delta_\lambda d\lambda$) is not necessarily zero. To put it loosely, radiative equilibrium does not hold for the separate wave-lengths independently.

### § 3. *The Theoretical Spectrum for Non-selective Absorption.*

We proceed to determine the variations of $I_\lambda$ and $I_\lambda'$ with $\lambda$, more particularly the structure of the emergent radiation $I_\lambda(0)$, for the case where $k_\lambda$ is independent of $\lambda$. Writing $k$ for $k_\lambda$, and putting as usual

$$\int_0^x k\rho dx = \tau,$$

equations (1) become

$$\frac{1}{2}\frac{dI_\lambda}{d\tau} = I_\lambda - B_\lambda, \qquad \frac{1}{2}\frac{dI_\lambda'}{d\tau} = B_\lambda - I_\lambda' \quad . \quad . \quad (5)$$

We saw in § 3 of the preceding paper that the corresponding equations for the integrated radiation of all wave-lengths have for their solution

$$B = \frac{1}{2}F(1 + 2\tau) = B_0(1 + 2\tau) \quad . \quad . \quad . \quad (6)$$

$$I = B + B_0, \qquad I' = B - B_0 \quad . \quad . \quad . \quad (7)$$

Since B is now known, so is $B_\lambda$ in (5); using (6) to eliminate $\tau$ from (5), we have

$$\frac{dI_\lambda}{dB} = \frac{I_\lambda - B_\lambda}{B_0}, \qquad \frac{dI_\lambda'}{dB} = \frac{B_\lambda - I_\lambda'}{B_0} \quad . \quad . \quad (8)$$

Solving these,

$$I_\lambda e^{-B/B_0} = -\frac{1}{B_0}\int_{B_0}^{B} B_\lambda e^{-B/B_0} dB + H_\lambda,$$

$$I_\lambda' e^{-B/B_0} = \frac{1}{B_0}\int_{B_0}^{B} B_\lambda e^{+B/B_0} dB + K_\lambda,$$

where $H_\lambda$, $K_\lambda$ are independent of the temperature. Since $I_\lambda' = 0$ for all $\lambda$ when $\tau = 0$, we have $K_\lambda = 0$. Moreover, as $B \rightarrow \infty$ in the interior, $I_\lambda/I_\lambda' \rightarrow 1$, for the difference between the inward and outward streams becomes negligible compared with the streams themselves. It is easily found that this gives

$$H_\lambda = \frac{1}{B_0}\int_{B_0}^{\infty} B_\lambda e^{-B/B_0} dB.$$

Thus finally

$$I_\lambda = \frac{1}{B_0} e^{B/B_0} \int_{B}^{\infty} B_\lambda e^{-B/B_0} dB \quad . \quad . \quad . \quad (9)$$

$$I_\lambda' = \frac{1}{B_0} e^{-B/B_0} \int_{B_0}^{B} B_\lambda e^{B/B_0} dB \quad . \quad . \quad . \quad (10)$$

(We may also put the boundary condition at infinity in the form $I_\lambda/B_\lambda \to 1$, the radiation becoming ultimately isotropic black-body radiation.)

Equations (9) and (10) exhibit the spectral distribution of the streams of radiation at any point as derived from the temperature of the neighbouring material, weighted with an exponential absorption factor. Since $B \propto T^4$, the integrals are practically integrals with respect to the temperature. For the stream of radiation escaping at the boundary we have, on setting $B/B_0 = y$,

$$I_\lambda(0) = e\int_1^{\infty} B_\lambda e^{-y} dy \quad . \quad . \quad . \quad (11)$$

Now in the usual notation

$$B_\lambda \propto \lambda^{-5}(e^{hc/\lambda RT} - 1)^{-1},$$

or, putting $hc/\lambda RT_0 = a$, $\quad T = T_0 y^{\frac{1}{4}}$,

$$B_\lambda \propto \lambda^{-5}\left(e^{ay^{-\frac{1}{4}}} - 1\right)^{-1}.$$

Thus finally *the spectral distribution of the radiation emergent at the boundary, in radiative equilibrium, is given by*

$$I_\lambda(0) \propto \lambda^{-5}\int^{\infty} \frac{e^{-y}}{e^{ay^{-\frac{1}{4}}} - 1} dy \quad . \quad . \quad . \quad (12)$$

in which formula the only disposable constant is $T_0$, which is itself given by the *total* radiation. In particular the coefficient of absorption $k$ has disappeared.

Let us compare this spectrum with that calculated on the assumption that the star was a perfect radiator at its own effective temperature $T_1$. We should have in this case, since $T_1 = 2^{\frac{1}{4}}T_0$,

$$\begin{aligned} I_\lambda(0) &\propto \lambda^{-5}(e^{hc/\lambda RT_1} - 1)^{-1} \\ &\propto \lambda^{-5}(e^{2^{-\frac{1}{4}}a} - 1)^{-1} \end{aligned} \quad . \quad . \quad . \quad (13)$$

Writing (12) in the form

$$I_\lambda(0) \propto \lambda^{-5}\int_0^{\infty} \frac{e^{-y}}{e^{a(1+y)^{-\frac{1}{4}}} - 1} dy \quad . \quad . \quad (14)$$

we see that the two distributions could only be identical if, considering the integrand in (14) as the product of a certain function $f(y)$ and an exponential weighting factor $e^{-y}$, the weighted mean value of $f(y)$ were $f(1)$ for all values of $a$. This is in general not true. The function

$$\phi_1(a) = a^5\int_0^{\infty} \frac{e^{-y}}{e^{a(1+y)^{-\frac{1}{4}}} - 1} dy$$

can be evaluated by a power series (P) in $a$ when $a < 2\pi$, and by an asymptotic formula (A) when $a$ is large.* Now the maximum ordinate in the spectrum is found to occur in the neighbourhood of $a = 6$, and it is such values of $a$ that we are chiefly interested in; unfortunately, for such values of $a$ both expansions (P) and

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* See Appendix at end.

1921MNRAS..81..375M

(A) tend to become intractable, and it is a matter of some delicacy to secure agreement in the overlapping values. For this reason values from both sources are given in the subjoined table. For comparison the values of

$$\phi_0(\alpha) = \alpha^5(e^{2^{-\frac{1}{4}}\alpha} - 1)^{-1}$$

are also tabulated. The limiting values are found to be given by

$$\phi_1(\alpha) \sim 1\cdot167\alpha^4, \qquad \phi_0(\alpha) \sim 1\cdot189\alpha^4$$

for $\alpha$ very small, and by

$$\phi_1(\alpha) \sim \sqrt{\frac{8\pi}{5}}\left(\frac{\alpha}{4}\right)^{\frac{2}{5}} e^{-5(\alpha/4)^{\frac{4}{5}}}, \qquad \phi_0(\alpha) \sim \alpha^5 e^{-2^{-\frac{1}{4}}\alpha}$$

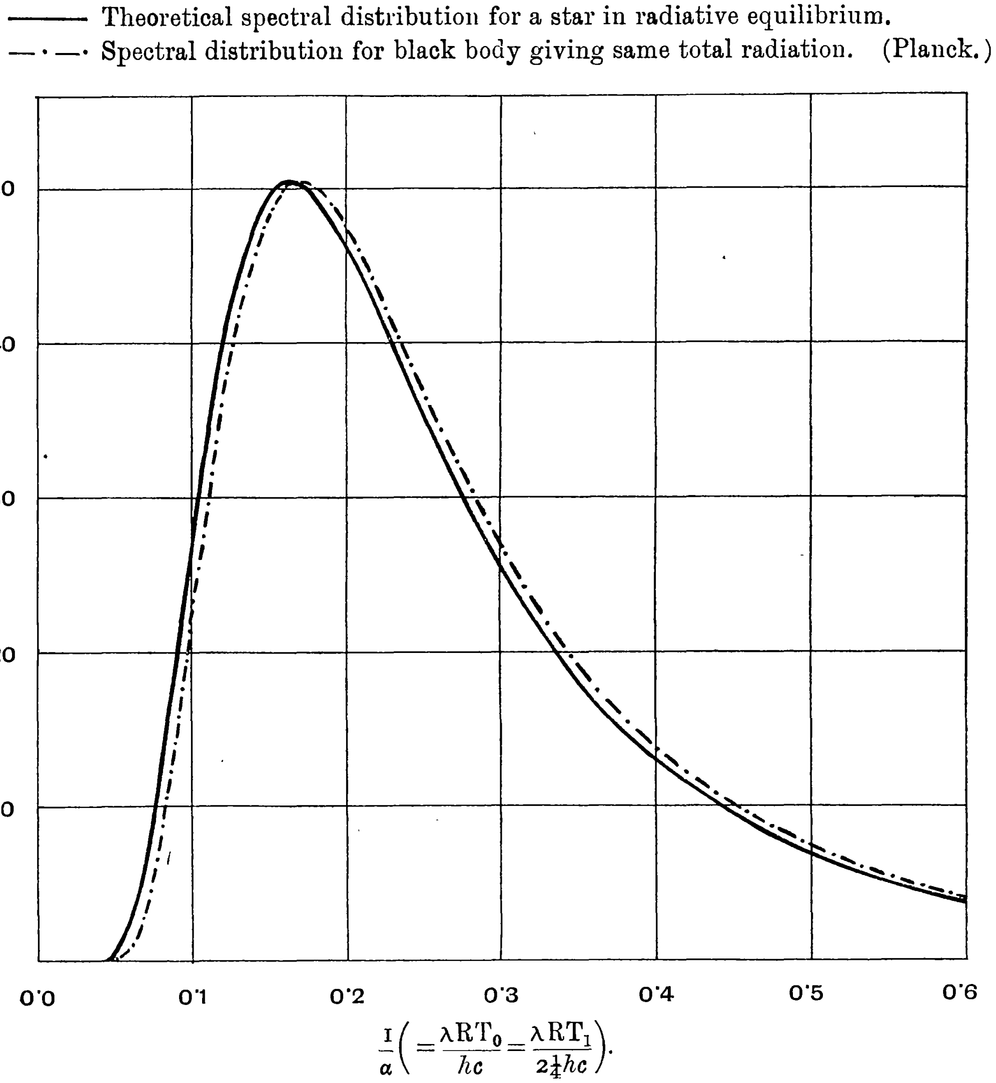
for $\alpha$ very large. Thus $\phi_1/\phi_0 \rightarrow 0\cdot982$, as $\alpha \rightarrow 0$ (large wave-lengths), and $\phi_1/\phi_0 \rightarrow \infty$ as $\alpha \rightarrow \infty$ (small wave-lengths).

TABLE I.

| $\alpha$. | $1/\alpha$. | $\phi_1(\alpha)$. (P). | $\phi_1(\alpha)$. (A). | $\phi_0(\alpha)$. |
|---|---|---|---|---|
| 1·0 | 1·0000 | 0·7386 | | 0·7585 |
| 1·5 | ·6667 | 2·912 | | 3·001 |
| 2·0 | ·5000 | 7·076 | | 7·314 |
| 2·5 | ·4000 | 13·124 | | 13·593 |
| 3 | ·3333 | 20·456 | | 21·200 |
| 3·5 | ·2857 | 28·218 | | 29·317 |
| 4 | ·2500 | 35·551 | 35·56 | 36·712 |
| 4·5 | ·2222 | 41·758 | 41·76 | 42·920 |
| 5 | ·2000 | 46·388 | 46·37 | 47·359 |
| 5·5 | ·1818 | 49·259 | 49·23 | 49·829 |
| 5·8 | ·1724 | | | 50·388 |
| 5·904 | ·1694 | | | 50·425 |
| 6 | ·1667 | | 50·36 | 50·395 |
| 6·1 | ·1639 | | 50·39 | |
| 6·2 | ·1613 | | 50·35 | |
| 6·5 | ·1538 | | 49·91 | 49·276 |
| 7 | ·1429 | | 48·17 | 46·808 |
| 8 | ·1250 | | 42·04 | 39·300 |
| 9 | ·1111 | | 34·24 | 30·526 |
| 10 | ·1000 | | 26·45 | 22·291 |
| 11 | ·0909 | | 19·62 | 15·483 |
| 12 | ·0833 | | 14·08 | 10·317 |
| 14 | ·0714 | | 6·67 | 4·148 |
| 16 | ·0625 | | 3·00 | 1·505 |
| 20 | ·0500 | | 0·517 | 0·1589 |

1921MNRAS..81..375M

The curves of $\phi_1$ and $\phi_0$ are shown in fig. 1 plotted against $1/\alpha$ (since $\alpha \propto 1/\lambda$) to give a "normal" spectrum. It appears that, in spite of their different analytical expressions, the functions closely resemble one another, the curve for radiative equilibrium being approximately a copy of the Planck curve *displaced to the violet.*



The maximum ordinate of the Planck curve occurs, as is well known, for

$$2^{-\frac{1}{4}}\alpha = 4{\cdot}965, \quad \text{or} \quad \alpha = 5{\cdot}904,$$

whilst detailed interpolation amongst the values in Table I. shows that the maximum ordinate of the new curve occurs for

$$\alpha = 6{\cdot}09,$$

the ratio of the two being 1·031. Noticing that $\alpha \propto 1/\lambda T_0 \propto 1/\lambda T_1$, and recalling that Wien's law is expressed in the form

$$\lambda_{\text{max.}}\,T = \text{constant},$$

1921MNRAS..81..375M

we may state the difference between the curves in the form : *the form of the normal spectrum in radiative equilibrium is such that the constant occurring in Wien's law* ($\lambda_{\max.}T_1$ = *constant*) *must be taken* 3·1 *per cent. smaller than for a perfect radiator,* $T_1$ *being the effective temperature of the star as deduced from the total radiation.* In other words, *we may expect the apparent temperature deduced from the total radiation to be* 3·1 *per cent. smaller than that deduced from an application of the usual form of Wien's law.* Since in our treatment we have assumed a "linear" or "tubular" flow of radiation, equivalent to averaging over all directions, we may expect the result just obtained to apply not to any particular portion of a stellar disc, but to the integrated radiation from the whole disc.

In the case of the sun it is well known that the temperature deduced from the wave-length of maximum intensity exceeds that deduced from the solar constant, though this is often vaguely attributed to the "imperfect radiating power of the sun." The temperature deduced by Abbot * from the wave-length of maximum energy for the whole disc is 6230°, whilst that deduced from the solar constant is 5860° ; this is about 6 per cent. lower, so that our correction accounts for about half of the discrepancy. But it is undesirable to stress this comparison with observation, owing to the notable departure of the solar spectrum from a black-body spectrum in other respects.

## § 4. *Variation of Colour over the Disc.*

The equation for the oblique radiation of wave-length $\lambda$ is

$$\cos\theta\frac{dI_\lambda}{d\tau} = I_\lambda - B_\lambda \quad . \quad . \quad . \quad (15)$$

$\theta$ being the inclination to the outward normal, and the solution of this for the outward and inward streams, with the proper boundary conditions, is

$$I_\lambda = e^{\tau\sec\theta}\int_\tau^\infty B_\lambda(\tau)\sec\theta\, e^{-\tau\sec\theta}d\tau \quad . \quad . \quad (16)$$

$$I_\lambda' = e^{-\tau\sec\psi}\int_0^\tau B_\lambda(\tau)\sec\psi\, e^{\tau\sec\psi}d\tau \quad . \quad . \quad (17)$$

where we have put $\pi - \theta = \psi$, $I_\lambda(\tau, \theta) = I_\lambda'(\tau, \psi)$ when $\frac{1}{2}\pi < \theta \leqslant \pi$. Changing the variable of integration in (16) to $\tau\sec\theta$ and putting $\tau = 0$, we have for the spectral distribution of the radiation escaping at the boundary

$$I_\lambda(0, \theta) = \int_0^\infty B_\lambda(\tau\cos\theta)e^{-\tau}d\tau \quad . \quad . \quad . \quad (18)$$

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* Abbot, *The Sun* (1912), p. 110 ; *Annals Astrophys. Obs. Smithson. Instit.*, iii. (1913), p. 200.

1921MNRAS..81..375M

or finally

$$I_\lambda(0,\ \theta) \propto \lambda^{-5} \int_0^\infty \frac{e^{-\tau} d\tau}{e^{\alpha(B(\tau\cos\theta)/B_0)^{-\frac{1}{4}}} - 1} \quad . \quad . \quad (19)$$

where $\alpha = hc/\lambda RT_0$ as before. Now the *total* amount of radiation emergent in direction $\theta$ is

$$I(0,\ \theta) = \int_0^\infty B(\tau\cos\theta) e^{-\tau} d\tau \quad . \quad . \quad . \quad (20)$$

which would itself correspond to an effective black-body temperature $T_1(\theta)$ given by

$$\left(\frac{T_1(\theta)}{T_0}\right)^4 = \frac{I(0,\ \theta)}{B_0};$$

and hence the spectral distribution of the radiation corresponding to its effective temperature would be

$$I_\lambda(0,\ \theta) \propto \frac{\lambda^{-5}}{e^{\alpha(I(0,\ \theta)/B_0)^{-\frac{1}{4}}} - 1} \quad . \quad . \quad . \quad (21)$$

Thus the spectrum of the radiation is not the same as the black-body spectrum corresponding to its effective temperature; the difference between (19) and (21) arises from the fact that the exponentially weighted mean value of $B(\tau\cos\theta)$ given by (20) (required in (21)) is not the same as the mean value of $B(\tau\cos\theta)$ given by (19).

To evaluate the spectrum further, we require the explicit form of the temperature distribution $B(\tau)$. It was shown in § 5 of the preceding paper that a fairly satisfactory first approximation is given by

$$B(\tau) = \tfrac{3}{8}F(\tfrac{4}{3} + 2\tau) = B_0(1 + \tfrac{3}{2}\tau) \quad . \quad . \quad . \quad (22)$$

and that this gives a coefficient of darkening $\frac{3}{5}$, in good agreement with the results of further approximations. Adopting this, we have from (19)

$$I_\lambda(0,\ \theta) \propto \lambda^{-5} \int_0^\infty \frac{e^{-\tau} d\tau}{e^{\alpha(1+\frac{3}{2}\tau\cos\theta)^{-\frac{1}{4}}} - 1} \quad . \quad . \quad (23)$$

whilst the effective temperature of the radiation would correspond to a black-body spectrum (from (20) and (21))

$$I_\lambda(0,\ \theta) \propto \frac{\lambda^{-5}}{e^{\alpha(1+\frac{3}{2}\cos\theta)^{-\frac{1}{4}}} - 1} \quad . \quad . \quad . \quad (24)$$

The two would only agree if the mean value of $\tau$ in (23) were unity.

It is now clear that the investigation of the radiation from the whole disc, given in § 3, really gives the case $\cos\theta = \frac{2}{3}$ ($\sin\theta = \cdot745$). Using the asymptotic formula for (23) for values of $\alpha$ in the neighbourhood of 6·5, it was found by constructing a portion of a table

1921MNRAS..81..375M

similar to Table I. that for $\cos\theta = 1$ the position of maximum energy occurs for $\alpha = 6{\cdot}51$; that for (24) occurs for

$$\alpha(1+\tfrac{3}{2})^{-\frac{1}{4}} = 4{\cdot}965, \quad \text{or} \quad \alpha = 6{\cdot}243,$$

and the ratio of these is 1·043. Thus the radiation from the centre of the disc has its wave-length of maximum intensity shifted 4·3 per cent. to the violet as compared with the black-body spectrum of the same effective temperature.

For other values of $\theta$, to avoid the labour of computing (23) an interpolation method was used. If we write

$$\int_0^\infty \frac{e^{-x}dx}{e^{\alpha(1+px)^{-\frac{1}{4}}} - 1} = \frac{1}{e^{\alpha(1+p\bar{x})^{-\frac{1}{4}}} - 1} \quad . \quad . \quad . \quad (25)$$

then $\bar{x} \rightarrow 1$ as $p \rightarrow 0$, for all values of $\alpha$; $\bar{x}$ is a function of $\alpha$, but fig. 1 suggests that it changes very slowly with $\alpha$. Assuming it to be constant, our previous work shows that for $p = 1$,

$$6{\cdot}09(1+\bar{x})^{-\frac{1}{4}} = 4{\cdot}965, \quad \text{or} \quad \bar{x} = 1{\cdot}26,$$

and for $p = \frac{3}{2}$,

$$6{\cdot}51(1+\tfrac{3}{2}\bar{x})^{-\frac{1}{4}} = 4{\cdot}965, \quad \text{or} \quad \bar{x} = 1{\cdot}31.$$

Interpolating between the values for $p = 0$, $1$, $\frac{3}{2}$, and using the results in (25), we have Table II. The column headed "percentage" gives the percentage displacement of the position of maximum intensity to the violet as compared with the black-body spectrum of the same effective temperature.

Table II.

| Cos $\theta$. | Sin $\theta$. | Percentage. |
|---|---|---|
| 1·0 | 0 | 4·3 |
| 0·9 | 0·436 | 4·1 |
| 0·8 | 0·600 | 3·7 |
| 0·6 | 0·800 | 2·8 |
| 0·4 | 0·915 | 1·7 |
| 0·2 | 0·979 | 0·6 |
| 0·0 | 1·000 | 0·0 |

The mean value of these percentages, averaged over the surface of the disc and weighted according to the relative brightness, is 3·1, confirming the result obtained in § 3.

Our final conclusion is, then, that *for a star in radiative equilibrium the wave-length of maximum energy for any point of the disc should be displaced to the violet, relatively to the position indicated by the effective temperature at that point, by an amount which varies from zero at the limb to about* 4 *per cent. at the centre, the average for the whole disc being about* 3 *per cent.* This shift to the violet as we go from limb to centre is in addition, it should be remembered, to that arising from the relative brightness of limb and centre.

### § 5. *Comparison with Observation.*

Table III., the data of which are taken from a similar table by Abbot, gives the variation of colour and temperature over the sun's disc; column I. gives the temperature deduced from the observed wave-length of maximum intensity, using Wien's law; column II. gives the effective temperature corresponding to the total intensity, using Stefan's law and Abbot's value of the solar constant; column III. gives the percentage excess of I. over II., and column IV. the calculated excess taken from Table II.

Table III.

| Sin $\theta$. | Cos $\theta$. | $\lambda_{\max}$. | I. | II. | III. | IV. |
|---|---|---|---|---|---|---|
| 0·00 | 1·000 | 0·458$\mu$ | 6400° | 6160° | 3·9 | 4·3 |
| 0·55 | 0·835 | 0·471 | 6220 | 6030 | 3·1 | 3·8 |
| 0·825 | 0·565 | 0·483 | 6070 | 5770 | 5·2 | 2·6 |
| 0·95 | 0·312 | 0·505 | 5800 | 5460 | 6·2 | 1·2 |

There is some agreement near the centre of the disc, but it is apparent that on the whole observation and theory are discordant. The percentages in columns III. and IV. change in opposite directions, the difference (III.–IV.) starting negative but changing sign and on the whole steadily increasing. The temperatures deduced from the wave-lengths do not decrease towards the limb so fast as is indicated by the total radiation. It is perhaps risky to draw conclusions merely from the position of maximum intensity, seeing that the sun's continuous spectrum is so unlike that of a black body, but study of the change of distribution, towards the limb, of the whole spectrum confirms the result. The centre of the disc appears to be slightly redder than would be expected, and the limb bluer; the centre-limb contrast in the shorter wave-lengths is too little.

This might be accounted * for by ignoring the *intensity* of the radiation and assuming a much higher value of $T_0$; this decreases $a$ and consequently decreases the centre-limb contrast for any given wave-length (relatively to a higher value of $T_0$ the bluer wave-lengths occupy the rôle of the redder ones and the calculated contrast is therefore smaller). It might also be accounted for by consideration of the actual form of the solar spectrum, taking into account the varying selective absorption. Lastly, it might be explained as a result of scattering. Since scattering is greater for blue light than for red, the blue light from the centre of the disc will come effectively from layers nearer the surface and therefore cooler than the red light, and consequently the centre of the disc will appear redder than it would otherwise. This has been pointed out by many writers; but it does not appear to have been mentioned explicitly that as the centre is poorer in blue rays

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* As is done by Lindblad.

1921MNRAS..81..375M

owing to scattering, so the limb should be relatively richer in blue rays for the same reason, for much of the blue light scattered from the radial beam from the interior will tend to emerge tangentially near the limb. The state of affairs at the limb is similar to that which causes the skylight in our own atmosphere to be blue (especially at right angles to the sun), as Rayleigh showed. It must, however, be noted further that scattering necessitates an increased temperature gradient (for a given flux) to drive the net flux of energy against the increased resistance, and the effect of this near the centre of the disc will be partly to compensate for the reddening.

## *Appendix.*

When $a < 2\pi$, the integral

$$a^5\int_1^\infty \frac{e^{-x}dx}{e^{ax^{-\frac{1}{4}}}-1} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

may be evaluated by expanding in the power series

$$a^4\int_1^\infty\left(1-\tfrac{1}{2}ax^{-\frac{1}{4}}+\frac{B_1}{2\,!}a^2x^{-\frac{1}{2}}-\frac{B_2}{4\,!}a^4x^{-1}+\frac{B_4}{6\,!}a^6x^{-\frac{3}{2}}- \quad . \quad . \quad . \right)x^{\frac{1}{4}}e^{-x}dx \quad (27)$$

where $B_1$, $B_2$ . . . are Bernoulli's numbers, and integrating term by term. The coefficients involve integrals of the type

$$\int_1^\infty x^s e^{-x}dx\,,$$

which are cases of the incomplete Γ-function; the values for $s > 1$ may be made to depend on those for $s \leqslant 1$ by means of a reduction formula. Incomplete Γ-functions not being readily available, the values for $s = \frac{1}{4}$, $s = \frac{1}{2}$ were calculated from

$$\int_1^\infty x^s e^{-x}dx = \Gamma(1-s) - \int_0^1 x^{-s}\left(1-\frac{x}{1\,!}+\frac{x^2}{2\,!}- \quad . \quad . \quad . \quad \right)dx,$$

the latter integral being evaluated by term-by-term integration. In this way the integral $\int_1^\infty x^{-\frac{1}{4}s}e^{-s}ds$ was calculated for $s = -1$, 0, 1, 3, . . . 39, and with these values and the values of Bernoulli's numbers given by Adams,* series (27) was evaluated up to $a = 5{\cdot}5$; for this it was found necessary to go to 40 terms to obtain the result correct to four significant figures, though the last few terms could be estimated by extrapolation.

For large values of $a$, we consider the more general integral

$$a^5e^{-1/p}\int_0^\infty \frac{e^{-x}dx}{e^{a(1+px)^{-\frac{1}{4}}}-1} \quad . \quad . \quad . \quad . \quad (28)$$

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* *Coll. Sci. Papers*, **1**, 455.

1921MNRAS..81..375M

required in § 4, which reduces to (26) when $p = 1$. Writing $1 + px = y$ and expanding the denominator, we have

$$\alpha^5 e^{-1/p}\int_0^\infty \frac{e^{-x}dx}{e^{(\alpha 1+px^{-\frac{1}{4}}} - 1} = \frac{1}{p}\alpha^5 \sum_{n=1}^{\infty}\int_1^\infty e^{-yp^{-1}-n\alpha y^{-\frac{1}{4}}}dy \quad . \quad (29)$$

which is rapidly convergent for large values of $\alpha$. The terms are all of the type

$$\frac{1}{p}\int_1^\infty e^{-xp^{-1}-\alpha x^{-\frac{1}{4}}}dx \quad . \quad . \quad . \quad (30)$$

for which we require an asymptotic expansion for large values of $\alpha$. Using a well-known method, we first notice that the minimum value of $xp^{-1} + \alpha x^{-\frac{1}{4}}$ occurs for $x = (\frac{1}{4}\alpha p)^{\frac{4}{5}}$ and is $5p^{-1}(\frac{1}{4}\alpha p)^{\frac{4}{5}}$, and we accordingly transform to a new variable $t$ given by

$$xp^{-1} + \alpha x^{-\frac{1}{4}} = t^2 + 5p^{-1}(\tfrac{1}{4}\alpha p)^{\frac{4}{5}},$$

so that $t = 0$ corresponds to $x = (\frac{1}{4}\alpha p)^{\frac{4}{5}}$. The integral (30) now becomes

$$\frac{1}{p}e^{-5p^{-1}(\frac{1}{4}\alpha p)^{\frac{4}{5}}}\int_{-\mathrm{N}}^\infty e^{-t^2}\frac{dx}{dt}dt \quad . \quad . \quad . \quad (31)$$

where

$$\mathrm{N}^2 = \alpha + p^{-1} - 5p^{-1}(\tfrac{1}{4}\alpha p)^{\frac{4}{5}},$$

and the positive value of N is taken when $\alpha p > 4$. For sufficiently small values of $t$, $x$ may be expanded in a power series in the form

$$x = (\tfrac{1}{4}\alpha p)^{\frac{4}{5}}(1 + a_1 t + a_2 t^2 + \ . \ . \ . \ ) \quad . \quad . \quad (32)$$

It is found that

$$a_1 = \left(\frac{8p}{5}\right)^{\frac{1}{2}}\left(\tfrac{1}{4}\alpha p\right)^{-\frac{2}{5}}, \qquad a_2 = \tfrac{3}{8}a_1^2, \qquad a_3 = \tfrac{3}{64}a_1^3,$$

$$a_4 = -\frac{3}{640}a_1^4, \qquad a_5 = \frac{39}{2 \times 64^2}a_1^5.$$

Most of the integral (31) is contributed by values of $t$ in the neighbourhood of zero, and in the usual manner an asymptotic expansion in powers of $(\frac{1}{4}\alpha p)^{-\frac{2}{5}}$ is obtained by inserting expansion (32) in (31) and integrating term by term. The result is

$$\frac{1}{p}\int_1^\infty e^{-xp^{-1}-\alpha x^{-\frac{1}{4}}}dx \sim \frac{8}{5a_1}e^{-8/a_1^2}\left[\tfrac{1}{2}\sqrt{\pi}\Big(1 + \Phi(\mathrm{N})\Big)\left(1 + \frac{9}{128}a_1^2\right)\right.$$

$$\left. + \tfrac{3}{8}a_1 e^{-\mathrm{N}^2}\left(1 - \tfrac{3}{16}\mathrm{N}a_1 - \frac{a_1^2}{40}(1 + \mathrm{N}^2)\right)\right] \quad . \quad . \quad (33)$$

as far as the term in $a_4$, $a_1$ being as above. Here $\Phi(\mathrm{N})$ denotes the error function

$$\Phi(\mathrm{N}) = \frac{2}{\sqrt{\pi}}\int_0^{\mathrm{N}} e^{-t^2}dt.$$

1921MNRAS..81..375M

The error in stopping at any particular power of $\alpha$ should be less than the last term retained. It was desirable to check expansions (27) and (29) against one another for such values of $\alpha$ as were possible (for $p=1$), though for values of $\alpha$ so small as 4 and 5, which were practically the only ones for which a comparison was possible, large errors would be expected in formula (33). It was found, however, that by stopping at the term in $\alpha^{-\frac{4}{6}}$ (term in $a_4$), even when the next term was numerically smaller, an unexpectedly good agreement between (27) and (29) was obtained; the figures have been given in Table I. From $\alpha=4$ to $\alpha=5{\cdot}5$ the error is never more than three units in the fourth figure, whilst if the next term had been retained the error would have been nearly a unit in the second figure. I am unable to account for this apparently fortuitous accuracy. In consequence of it expansion (33) was stopped at the term in $a_4$ for all the values of $\alpha$ considered (up to $\alpha=20$). The convergence of (29) is so rapid that two terms sufficed when $\alpha>5$.

It is a pleasant duty to thank Professor Newall for his continued interest in both the foregoing investigations, and for his care and suggestions in revision.

*Solar Physics Observatory, Cambridge:*
1921 *March* 1.

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## *Preliminary Values of Variation of Latitude at Greenwich during* 1920.

(*Communicated by the Astronomer Royal.*)

In the middle of the year 1919 the observing programme with the floating zenith telescope was modified, the faintest and brightest stars being removed from the list and other pairs inserted, so as to reduce the mean zenith distance of each group. From the observations during the period 1919·5 to 1921·0 preliminary places for the new pairs have been obtained and revised group corrections deduced. The new system of declinations has also been reduced to the former system.

The period in question has unfortunately been broken by several long spells of cloudy weather, so that the observations are not distributed with the uniformity that is desirable for the accurate determination of group corrections. Some of the corrections are consequently weakly determined at present.

With the extra material derived from a further year's observations, it is possible to improve the values given for the latitude variation for 1919 (*M.N.*, *R.A.S.*, **80**, 454, 1920). The following table gives the values of the latitude variation, in the sense mean *minus* observed latitude, during the years 1919–1920:—*

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* In *M.N.*, **80**, 454, for "observed *minus* mean latitude" read "mean *minus* observed latitude."